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Supergravity and a confining gauge theory: duality cascades and χSB-resolution of naked singularities

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Abstract: We revisit the singular IIB supergravity solution describing M-fractional 3-branes on the conifold [hep-th/0002159]. Its 5-form flux decreases, which we explain by showing that the relevant $\mathcal{N}=1$ SUSY $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$ gauge theory undergoes repeated Seiberg-duality transformations in which $N\to N-M$. Far in the IR the gauge theory confines; its chiral symmetry breaking removes the singularity of [hep-th/0002159] by deforming the conifold. We propose a non-singular pure-supergravity background dual to the field theory on all scales, with small curvature everywhere if the 't Hooft coupling g_sM is large. In the UV it approaches that of [hep-th/0002159], incorporating the logarithmic flow of couplings. In the IR the deformation of the conifold gives a geometrical realization of chiral symmetry breaking and confinement. We suggest that pure $\mathcal{N}=1$ Yang-Mills may be dual to strings propagating at small g_sM on a warped deformed conifold. We note also that the standard model itself may lie at the base of a duality cascade.

Keywords: Non-perturbative Effects, Confinement, Duality in Gauge Field Theories, AdS-CFT Correspondance.

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Contents

1. Introduction

A fruitful extension of the basic AdS/CFT correspondence [1, 2, 3] stems from studying branes at conical singularities [4]–[7]. Consider, for instance, a stack of D3-branes placed at the apex of a Ricci-flat 6-d cone Y_6 whose base is a 5-d Einstein manifold X_5. Comparing the metric with the D-brane description leads one to conjecture that type-IIB string theory on $AdS_5 \times X_5$ is dual to the low-energy limit of the world volume theory on the D3-branes at the singularity.

A useful example of this correspondence has been to study D3-branes on the conifold [6]. When the branes are placed at the singularity, the resulting $\mathcal{N} = 1$ superconformal field theory has gauge group $\mathrm{SU}(N) \times \mathrm{SU}(N)$. It contains chiral superfields A_1, A_2 transforming as $(\mathbf{N}, \overline{\mathbf{N}})$ and superfields B_1, B_2 transforming as $(\overline{\mathbf{N}}, \mathbf{N})$, with superpotential $\mathcal{W} = \lambda\epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}\, A_i B_k A_j B_l$. The two gauge couplings do not flow, and indeed can be varied continuously without ruining conformal invariance.

For many singular spaces Y_6 there are also fractional D3-branes which can exist only within the singularity [8, 9, 10, 11]. These fractional D3-branes are D5-branes wrapped over (collapsed) 2-cycles at the singularity. In the case of the conifold, the singularity is a point. The addition of M fractional branes at the singular point changes the gauge group to $\mathrm{SU}(N+M) \times \mathrm{SU}(N)$; the four chiral superfields remain, now in the representation $(\mathbf{N}+\mathbf{M}, \overline{\mathbf{N}})$ and its conjugate, as does the superpotential [10, 11]. The theory is no longer conformal. Instead, the relative gauge coupling $g_1^{-2} - g_2^{-2}$ runs logarithmically, as pointed out in [11], where the supergravity equations corresponding to this situation were solved to leading order in M/N. In [12] this solution was completed to all orders; the conifold suffers logarithmic warping, and the relative gauge coupling runs logarithmically at all scales. The D3-brane charge, i.e. the 5-form flux, decreases logarithmically as well. However, the logarithm in the solution is not cut off at small radius; the D3-brane charge eventually becomes negative and the metric becomes singular.

In [12] it was conjectured that this solution corresponds to a flow in which the gauge group factors repeatedly drop in size by M units, until finally the gauge groups are perhaps $\mathrm{SU}(2M) \times \mathrm{SU}(M)$ or simply $\mathrm{SU}(M)$. It was further suggested that the strong dynamics of this gauge theory would resolve the naked singularity in the metric. Here, we show that this conjecture is correct. The flow is in fact an infinite series of Seiberg duality transformations — a "duality cascade" — in which the number of colors repeatedly drops by M units. Once the number of colors in the smaller gauge group is fewer than M, non-perturbative effects become essential. We will show that these gauge theories have an exact anomaly-free $\mathbb{Z}_{2M}$ R-symmetry, which is broken dynamically, as in pure $\mathcal{N} = 1$ Yang-Mills theory, to $\mathbb{Z}_2$. In the supergravity, this occurs through the deformation of the conifold.[1] In short, the resolution of the naked singularity found in [12] occurs through the chiral symmetry breaking of the gauge theory. The resulting space, *a warped deformed conifold*, is completely non singular and without a horizon, leading to confinement. If the low-energy gauge theory has fundamental matter, a horizon appears and leads to screening.

2. Branes and fractional branes on the conifold

2.1 The conifold

The conifold is described by the following equation in $\mathbb{C}^4$:

$$\sum_{n=1}^{4} z_n^2 = 0\,. \tag{2.1}$$

[1]For a five-dimensional supergravity approach to chiral symmetry breaking, see [13].

Equivalently, using $z_{ij} = \frac{1}{\sqrt{2}} \sum_n \sigma^n_{ij} z_n$, where σ^n are the Pauli matrices for $n = 1, 2, 3$ and σ^4 is i times the unit matrix, it may be written as

$$\det_{i,j} z_{ij} = 0\,. \tag{2.2}$$

This is a cone whose base is a coset space $\mathbb{T}^{11} = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\,\mathrm{U}(1)$, with topology $S^2 \times S^3$ and symmetry group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. As discussed in [10], fractional D3-branes at the singularity $z_n = 0$ of the conifold are simply D5-branes which are wrapped on the S^2 of $\mathbb{T}^{11}$. The Einstein metric of $\mathbb{T}^{11}$ may be written down explicitly [14]:

$$ds^2_{\mathbb{T}^{11}} = \frac{1}{9}\left(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i\right)^2 + \frac{1}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right). \tag{2.3}$$

It will be useful to employ the following basis of 1-forms on the compact space [15]:

$$\begin{aligned} g^1 &= \frac{e^1 - e^3}{\sqrt{2}}\,, \qquad g^2 = \frac{e^2 - e^4}{\sqrt{2}}\,, \\ g^3 &= \frac{e^1 + e^3}{\sqrt{2}}\,, \qquad g^4 = \frac{e^2 + e^4}{\sqrt{2}}\,, \qquad g^5 = e^5\,, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} &e^1 \equiv -\sin\theta_1 d\phi_1\,, \qquad e^2 \equiv d\theta_1\,, \qquad e^3 \equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2\,, \\ &e^4 \equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2\,, \qquad e^5 \equiv d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\,. \end{aligned} \tag{2.5}$$

In terms of this basis, the Einstein metric on $\mathbb{T}^{11}$ assumes the form

$$ds^2_{\mathbb{T}^{11}} = \frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2\,, \tag{2.6}$$

and the metric on the conifold is

$$ds^2_6 = dr^2 + r^2 ds^2_{\mathbb{T}^{11}}\,. \tag{2.7}$$

2.2 The gauge theory

If we place N D3-branes and M fractional D3-branes on the conifold, we obtain an $\mathrm{SU}(N+M) \times \mathrm{SU}(N)$ gauge group. The two gauge group factors have holomorphic scales Λ_1 and $\tilde{\Lambda}_1$. The matter consists of two chiral superfields A_1, A_2 in the $(\mathbf{N+M}, \overline{\mathbf{N}})$ representation and two fields B_1, B_2 in the $(\overline{\mathbf{N+M}}, \mathbf{N})$ representation. The superpotential of the model is

$$W = \lambda_1 \operatorname{tr}\,(A_i B_j A_k B_\ell)\epsilon^{ik}\epsilon^{j\ell}\,. \tag{2.8}$$

	$\mathrm{SU}(N_+)$	$\mathrm{SU}(N)$	$\mathrm{SU}(2)$	$\mathrm{SU}(2)$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_R$
A_1, A_2	$\mathbf{N_+}$	$\overline{\mathbf{N}}$	$\mathbf{2}$	$\mathbf{1}$	$\frac{1}{2N_+N}$	$\frac{1}{2N_+N}$	$\frac{1}{2}$
B_1, B_2	$\overline{\mathbf{N_+}}$	$\mathbf{N}$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2N_+N}$	$\frac{1}{2N_+N}$	$\frac{1}{2}$
$\Lambda_1^{3N_+-2N}$					0	$\frac{2}{N_+}$	$2M$
$\tilde{\Lambda}_1^{3N-2N_+}$					0	$\frac{2}{N}$	$-2M$
λ_1					0	$-\frac{2}{N_+N}$	0

Table 1: Quantum numbers in the $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$ model; we have written $N_+\equiv N+M$ for concision.

The model has a $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ global symmetry; the first (second) factor rotates the flavor index of the A_i (B_i), while the "baryon" U(1) sends $A_i\to A_ie^{i\alpha}$, $B_i\to B_ie^{-i\alpha}$.[2] There are also two spurious U(1) transformations, one an R-symmetry and one a simple axial symmetry, under which λ_1, Λ_1 and $\tilde{\Lambda}_1$ are generally not invariant. The charges of the matter and the couplings under the symmetries (excepting the SU(2) flavor symmetries) are given in table 1. Although $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_R$ are anomalous, there is a discrete $\mathbb{Z}_{2M}$ R-symmetry under which the theory is invariant. In particular, if we let

$$[A_i, B_j]\longrightarrow [A_i, B_j]e^{2\pi in/4M}\,,\quad n=1,2,\ldots,2M\,, \tag{2.9}$$

and rotate the gluinos by $e^{2\pi in/2M}$, then the superpotential rotates by $e^{2\pi in/M}$ with λ_1, Λ_1 and $\tilde{\Lambda}_1$ unchanged.

The classical field theory is well aware that it represents branes moving on a conifold [6, 7]. Let us consider the case where the A_i and B_k have diagonal expectation values, $\langle A_i\rangle=\mathrm{diag}(a_i^{(1)},\ldots,a_i^{(N)})$, $\langle B_i\rangle=\mathrm{diag}(b_i^{(1)},\ldots,b_i^{(N)})$. The F-term equations for a supersymmetric vacuum

$$B_1A_iB_2-B_2A_iB_1=0\,,\qquad A_1B_kA_2-A_2B_kA_1=0 \tag{2.10}$$

are automatically satisfied in this case, while the D-term equations require $|a_1^{(r)}|^2+|a_2^{(r)}|^2-|b_1^{(r)}|^2-|b_2^{(r)}|^2=0$. Along with the phases removed by the maximum abelian subgroup of the gauge theory, the D-terms leave only $3N$ independent complex variables. Define $n_{ik}^{(r)}=a_i^{(r)}b_k^{(r)}$; then the D-term and gauge invariance conditions are satisfied by using the $n_{ik}^{(r)}$ as coordinates. These $4N$ complex coordinates satisfy the condition, for each r,

$$\det_{i,k} n_{ik}^{(r)}=0\,. \tag{2.11}$$

This is the same as eq. (2.2). Thus, for each $r=1,\ldots,N$, the coordinates $n_{11}^{(r)}$, $n_{12}^{(r)}$, $n_{21}^{(r)}$, $n_{22}^{(r)}$, are naturally thought of as the position of a D3-brane moving on a conifold.

[2] The question of whether this U(1) is actually gauged is subtle. We believe that it is global, and arguments for this were given in [6, 7, 16].

There are various combinations of the fields and parameters which are invariant under the global symmetries. One is

$$I_1 \sim \lambda_1^{3M} \frac{\tilde{\Lambda}_1^{3N-2(N+M)}}{\Lambda_1^{3(N+M)-2N}} \left[\mathrm{tr}(A_i B_j A_k B_\ell \epsilon^{ik} \epsilon^{j\ell})\right]^{2M} . \tag{2.12}$$

In addition, there are simple invariants such as

$$R_1^{(1)} = \frac{\mathrm{tr}\ [A_i B_j]\ \mathrm{tr}[A_k B_\ell] \epsilon^{ik} \epsilon^{j\ell}}{\mathrm{tr}\ (A_i B_j A_k B_\ell \epsilon^{ik} \epsilon^{j\ell})} ; \tag{2.13}$$

there are many other similar invariants, in each of which the same number of A and B fields appear in numerator and denominator but with color and flavor indices contracted differently. Finally there is a constant invariant

$$J_1 \equiv \lambda_1^{(N+M)+N} \Lambda_1^{3(N+M)-2N} \tilde{\Lambda}_1^{3N-2(N+M)} \tag{2.14}$$

which plays the role of a dimensionless complex coupling analogous to τ in $\mathcal{N}=4$ Yang-Mills.

The superpotential of the model will be renormalized and takes the general form

$$W = \lambda_1\ \mathrm{tr}\ (A_i B_j A_k B_\ell) \epsilon^{ik} \epsilon^{j\ell}\ F_1 \left(I_1, J_1, R_1^{(s)}\right) , \tag{2.15}$$

where F_1 is a function which we will not fully determine.

2.3 The conformal case: $M = 0$

If there are no fractional D3-branes, then the $\mathrm{U}(1)_R$ is anomaly-free, and the theory is superconformal (or if the couplings g_1, g_2, λ are chosen completely arbitrarily, it will flow until it becomes conformal in the infrared). There are two dimensionless global invariants $\lambda^2 \Lambda_1 \tilde{\Lambda}_1$, the overall coupling $\tau_1 + \tau_2$, and $\tilde{\Lambda}_1/\Lambda_1$, the relative coupling $\tau_1 - \tau_2$, which are built purely from the parameters and may be chosen arbitrarily. Thus [17, 18, 6] there are two exactly marginal operators in the theory which preserve the continuous global symmetries. (There are other marginal operators which partially preserve these symmetries.) This was the case studied in [6], where it was shown the supergravity dual of this field theory is simply $AdS_5 \times \mathbb{T}^{11}$.

In order to match the two couplings to the moduli of the type-IIB theory on $AdS_5 \times \mathbb{T}^{11}$, one notes that the integrals over the S^2 of $\mathbb{T}^{11}$ of the NS-NS and R-R 2-form potentials, B_2 and C_2, are moduli. In particular, the two gauge couplings are determined as follows [6, 7]:

$$\frac{1}{g_1^2} + \frac{1}{g_2^2} \sim e^{-\phi} , \tag{2.16}$$

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim e^{-\phi} \left[\left(\int_{S^2} B_2\right) - 1/2\right] , \tag{2.17}$$

where $(\int_{S^2} B_2)$ is normalized in such a way that its period is equal to 1.[3] The matching between the moduli is one of the simplest checks of the duality. It is further possible to build a detailed correspondence between various gauge invariant operators in the $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory and modes of the type-IIB theory on $AdS_5 \times \mathbb{T}^{11}$ [6, 19, 20].

In [21, 22], it was noted that there exists a type-IIA construction [23] which is T-dual to N D3-branes at the conifold. It involves two NS5-branes: one oriented in the (12345) plane, and the other in the (12389) plane. The coordinate x^6 is compactified on a circle of circumference l_6, and there are N (1236) D4-branes wrapped around the circle. If the NS5-branes were parallel, then the low-energy field theory would be the $\mathcal{N} = 2$ supersymmetric $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory with bifundamental matter (this type-IIA configuration is T-dual to N D3-branes at the $\mathcal{N} = 2$ $\mathbb{Z}_2$ orbifold singularity). Turning on equal and opposite masses for the two adjoint chiral superfields corresponds to rotating one of the NS5-branes. Under this relevant deformation the $\mathbb{Z}_2$ orbifold field theory flows to the $\mathcal{N} = 1$ supersymmetric conifold field theory [6, 22].

In terms of the type-IIA brane construction, the two gauge couplings are determined by the positions of the NS5-branes along the x^6 circle. If one of the NS5-branes is located at $x_6 = 0$ and the other at $x_6 = a$, then [22]

$$\frac{1}{g_1^2} = \frac{l_6 - a}{g_s}\,, \qquad \frac{1}{g_2^2} = \frac{a}{g_s}\,. \tag{2.18}$$

The couplings are equal when the NS5-branes are located diametrically opposite each other (in the type-IIB language this corresponds to $\int_{S^2} B_2$ being equal to half of its period). As the NS5-branes approach each other, one of the couplings becomes strong. This simple geometrical picture will be useful for analyzing the RG flows in the following sections.

2.4 The RG cascade: $M > 0$

Now let us consider the effect of adding M fractional D3-branes, which as shown in [10] corresponds to wrapping M D5-branes over the S^2 of $\mathbb{T}^{11}$. The D5-branes serve as sources of the magnetic RR 3-form flux through the S^3 of $\mathbb{T}^{11}$. Therefore, the supergravity dual of this field theory involves M units of the 3-form flux, in addition to N units of the 5-form flux:

$$\int_{S^3} F_3 = M\,, \qquad \int_{\mathbb{T}^{11}} F_5 = N\,. \tag{2.19}$$

[3] These equations are crucial for relating the SUGRA background to the field theory beta functions when the theory is generalized to $\mathrm{SU}(N+M) \times \mathrm{SU}(N)$ [11, 12].

In the SUGRA description the 3-form flux is the source of conformal symmetry breaking. Indeed, now B_2 cannot be kept constant and acquires a radial dependence [11]:

$$\int_{S^2} B_2 \sim M e^{\phi} \ln(r/r_0)\,, \tag{2.20}$$

while the dilaton stays constant at least to linear order in M. Since the AdS_5 radial coordinate r is dual to the RG scale [1, 2, 3], (2.17) implies a logarithmic running of $1/g_1^2 - 1/g_2^2$ in the $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$ gauge theory. This is in accord with the exact β-functions:

$$\frac{d}{d\log(\Lambda/\mu)}\frac{8\pi^2}{g_1^2} \sim 3(N+M) - 2N(1-\gamma)\,, \tag{2.21}$$

$$\frac{d}{d\log(\Lambda/\mu)}\frac{8\pi^2}{g_2^2} \sim 3N - 2(N+M)(1-\gamma)\,, \tag{2.22}$$

where γ is the anomalous dimension of operators $\mathrm{Tr}A_iB_j$. A priori, the conformal invariance of the field theory for $M=0$ requires that $\gamma = -1/2 + O(M/N)$. Taking the difference of the two equations in (2.21) we then find

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} \sim M\ln(\Lambda/\mu)\Big[3 + 2(1-\gamma)\Big]\,, \tag{2.23}$$

in agreement with (2.20) found on the SUGRA side. The constancy of the dilaton ϕ to order M is consistent with the field theory only if $\gamma = -1/2 + O[(M/N)^2]$. Fortunately, the field theory in table 1 has an obvious symmetry $M \to -M, N \to N+M$, which to leading order in M/N is $M \to -M$ with N fixed. Clearly γ is even under this symmetry and so cannot depend on M/N at first order.

The SUGRA analysis of [11] was carried out to the linear order in M/N. Luckily, it is possible to construct an exact solution taking into account the back-reaction of H_3 and F_3 on other fields [12]. In this solution $e^{\phi} = g_s$ is exactly constant, which translates into the vanishing of the β-function for $1/g_1^2 + 1/g_2^2$ in the dual field theory. As in [11],[4]

$$F_3 = M\omega_3\,, \qquad B_2 = 3g_sM\omega_2\ln(r/r_0)\,, \tag{2.24}$$

$$H_3 = dB_2 = 3g_sM\frac{1}{r}dr\wedge\omega_2\,, \tag{2.25}$$

where

$$\omega_2 = \frac{1}{2}\left(g^1\wedge g^2 + g^3\wedge g^4\right) = \frac{1}{2}\left(\sin\theta_1 d\theta_1\wedge d\phi_1 - \sin\theta_2 d\theta_2\wedge d\phi_2\right), \tag{2.26}$$

$$\omega_3 = \frac{1}{2}g^5\wedge\left(g^1\wedge g^2 + g^3\wedge g^4\right). \tag{2.27}$$

[4]We are not keeping track of the overall factor multiplying M, which is determined by the flux quantization.

The relative factor of 3 in (2.24), which is related to the coefficients in the metric (2.3), appears to be related to the factor of 3 in the $\mathcal{N}=1$ beta function (2.23). This gives the correct value of beta function from a purely geometrical point of view.

Both ω_2 and ω_3 are closed. Note also that

$$g_s \star_6 F_3 = -H_3\,, \qquad g_s F_3 = \star_6 H_3\,, \tag{2.28}$$

where $\star_6$ is the Hodge dual with respect to the metric ds_6^2. Thus, the complex 3-form $g_s G_3 = g_s F_3 + iH_3$ satisfies the self-duality condition

$$\star_6 G_3 = iG_3\,. \tag{2.29}$$

This is consistent with G_3 being either a $(0,3)$ form or a $(2,1)$ form on the conifold. The Calabi-Yau form carries $\mathrm{U}(1)_R$ charge equal to 2, while G_3 does not transform under the $\mathrm{U}(1)_R$. Hence, the only consistent possibility appears to be that G_3 is a harmonic $(2,1)$ form.[5]

It follows from eq. (2.28) that

$$g_s^2 F_3^2 = H_3^2\,, \tag{2.30}$$

which implies that the dilaton is constant, $\phi = 0$. Since $F_{3\mu\nu\lambda}H_3^{\mu\nu\lambda} = 0$, the RR scalar vanishes as well. The 10-d metric is

$$ds_{10}^2 = h^{-1/2}(r)dx_n dx_n + h^{1/2}(r)(dr^2 + r^2 ds_{\mathbb{T}^{11}}^2)\,, \tag{2.31}$$

where

$$h(r) = b_0 + 4\pi \frac{g_s N + a(g_s M)^2 \ln(r/r_0) + a(g_s M)^2/4}{r^4} \tag{2.32}$$

and a is a constant of order 1. Note that, for the ansatz (2.31), the solution for h may be determined from the trace of the Einstein equation:[6]

$$h^{-3/2}\nabla_6^2 h \sim g_s^2 F_3^2 + H_3^2 = 2g_s^2 F_3^2\,, \tag{2.33}$$

where ∇_6^2 is the laplacian on the conifold. Since $F_3^2 \sim M^2 r^{-6} h^{-3/2}$, the solution (2.32) follows directly.

An important feature of this background, which is not visible to linear order in M, is that F_5 acquires a radial dependence [12]. This is because

$$F_5 = dC_4 + B_2 \wedge F_3\,, \tag{2.34}$$

and $\omega_2 \wedge \omega_3 \sim \mathrm{vol}(\mathbb{T}^{11})$. Thus, we may write

$$F_5 = \mathcal{F}_5 + *\mathcal{F}_5\,, \qquad \mathcal{F}_5 = \mathcal{K}(r)\,\mathrm{vol}(\mathbb{T}^{11}) \tag{2.35}$$

[5]We are grateful to S. Gubser and E. Witten for discussions on this issue.
[6]We are grateful to A. Tseytlin for explaining this to us.

and

$$\mathcal{K}(r) = N + a g_s M^2 \ln(r/r_0)\,. \tag{2.36}$$

The novel phenomenon in this solution is that the 5-form flux present at the UV scale $r = r_0$ may completely disappear by the time we reach a scale $r = \tilde{r}$ where $\mathcal{K}(\tilde{r}) = 0$. This is related to the fact that the flux $\int_{S^2} B_2$ is not a periodic variable in the SUGRA solution: as this flux goes through a period, $\mathcal{K}(r) \to \mathcal{K}(r) - M$ which has the effect of decreasing the 5-form flux by M units. We will shortly relate this decrease, which we refer to for now as the "RG cascade", to Seiberg duality.

In order to eliminate the asymptotically flat region for large r we use the well-known device of setting $b_0 = 0$ (this corresponds to choosing the special solution of [12, section 5]). In terms of the scale $\tilde{r}$, we then have

$$\mathcal{K}(r) = a g_s M^2 \ln\left(\frac{r}{\tilde{r}}\right), \qquad h(r) = \frac{4\pi g_s}{r^4}\left[\mathcal{K}(r) + a g_s \frac{M^2}{4}\right]. \tag{2.37}$$

This solution has a naked singularity at $r = r_s$ where $h(r_s) = 0$. Writing

$$h(r) = \frac{L^4}{r^4}\ln\left(\frac{r}{r_s}\right), \qquad L^2 \sim g_s M\,, \tag{2.38}$$

we then have a purely logarithmic RG cascade:

$$ds^2 = \frac{r^2}{L^2\sqrt{\ln(r/r_s)}}dx_n dx_n + \frac{L^2\sqrt{\ln(r/r_s)}}{r^2}dr^2 + L^2\sqrt{\ln(r/r_s)}ds^2_{\mathbb{T}^{11}}\,. \tag{2.39}$$

This is essentially the metric of [12, section 5] expressed in terms of a different radial coordinate. Since $\mathbb{T}^{11}$ expands slowly toward large r, the curvatures decrease there so that corrections to the SUGRA are negligible. Therefore, there is no obstacle for using this solution as $r \to \infty$ where the 5-form flux diverges. The field theory explanation of the divergence is that the RG cascade goes on forever as the scale is increased, generating bigger and bigger N in the UV.

As the theory flows to the IR, the cascade must stop, however, because negative N is physically non sensical. Thus, we should not be able to continue the solution (2.39) to $r < \tilde{r}$ where $\mathcal{K}(r)$ is negative. The radius of $\mathbb{T}^{11}$ at $r = \tilde{r}$ is of order $\sqrt{g_s M}$. The gauge group at this scale is essentially SU(M), and it is satisfying to see the appearance of $g_s M$, which is the 't Hooft coupling. As usual, if the 't Hooft coupling is large then the SUGRA solution has small curvatures. Nevertheless, the fact that the solution of [12] is singular tells us that it has to be modified, at least in the IR. After understanding the RG cascade, we will study the dynamics of the corresponding field theory, and will see how this singularity is removed.

3. The $\mathcal{N} = 1$ RG cascade is a duality cascade

We now trace the jumps in the rank of the gauge group to a well-known phenomenon in the dual $\mathcal{N} = 1$ field theory, namely, Seiberg duality [24]. The essential observation is that $1/g_1^2$ and $1/g_2^2$ flow in opposite directions and, according to (2.21), there is a scale where the SU($N+M$) coupling, g_1, diverges. To continue past this infinite coupling, we perform a $\mathcal{N}=1$ duality transformation on this gauge group factor. The SU($N+M$) gauge factor has $2N$ flavors in the fundamental representation. Under a Seiberg duality transformation, this becomes an SU($2N-[N+M]$) = SU($N-M$) gauge group with $2N$ flavors, which we may call a_i and b_i, along with "meson" bilinears $M_{ij} = A_i B_j$. The fields a_i and b_i are fundamentals and antifundamentals of SU(N), while the mesons are in the adjoint-plus-singlet of SU(N). The superpotential after the transformation

$$W = \lambda_1 \,\mathrm{tr}\; M_{ij} M_{k\ell} \epsilon^{ik} \epsilon^{j\ell} F_1 \left(I_1, J_1, R_1^{(s)} \right) + \frac{1}{\mu} \mathrm{tr}\; M_{ij} a_i b_j \,, \tag{3.1}$$

where μ is the matching scale for the duality transformation [17], shows the M_{ij} are actually massive. We may integrate them out

$$0 = 2\lambda_1 M_{k\ell} \epsilon^{ik} \epsilon^{j\ell} F_1 \left(I_1, J_1, R_1^{(s)} \right) - \frac{1}{\mu} \mathrm{tr}\; a_i b_j \tag{3.2}$$

leaving a superpotential

$$W = \lambda_2 \,\mathrm{tr}\; a_i b_j a_k b_\ell \epsilon^{ik} \epsilon^{j\ell} F_2 \left(I_2, J_2, R_2^{(s)} \right). \tag{3.3}$$

Here F_2, λ_2, I_2, J_2 and R_2 are defined similarly as in the original theory. Thus we obtain an SU(N) $\times$ SU($N-M$) theory which resembles closely the theory we started with.[7]

Let us study the matching more carefully. We define, for reasons which will become clear in a moment, the strong coupling scale of the SU($N-M$) factor to be $\tilde{\Lambda}_2$. The strong coupling scale of the SU(N) factor is not the same as it was before the duality (since the number of flavors in the SU(N) gauge group has changed) and its old scale $\tilde{\Lambda}_1$ must be replaced with a new strong coupling scale Λ_2. The matching conditions relating these scales are of the form

$$\lambda_2 \propto \frac{1}{\mu^2 \lambda_1} \tag{3.4}$$

and

$$\Lambda_1^{3(N+M)-2N} \tilde{\Lambda}_2^{3(N-M)-2N} \propto \mu^{2N} \propto \lambda_1^M \tilde{\Lambda}_1^{3N-2(N+M)} \lambda_2^{-M} \Lambda_2^{3N-2(N-M)} \,. \tag{3.5}$$

It is easy to check that

$$I_2 \propto I_1 \qquad \text{and} \qquad J_2 \propto \frac{1}{J_1} \,. \tag{3.6}$$

[7]The fact that the quartic superpotential is left roughly invariant by the duality transformation in theories of this type has long been considered of interest. It was first noted in [17], where it was used to study duality in SO(3) gauge theories, and in [18], where its wider significance in Seiberg duality transformations was established.

	SU(N)	SU(N_-)	SU(2)	SU(2)	U(1)$_B$	U(1)$_A$	U(1)$_R$
a_1, a_2	$\overline{\mathbf{N}}$	$\mathbf{N}_-$	$\mathbf{2}$	$\mathbf{1}$	$\frac{1}{2NN_-}$	$\frac{1}{2NN_-}$	$\frac{1}{2}$
b_1, b_2	$\mathbf{N}$	$\overline{\mathbf{N}_-}$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2NN_-}$	$\frac{1}{2NN_-}$	$\frac{1}{2}$
$\Lambda_2^{3N-2N_-}$					0	$\frac{2}{N}$	$2M$
$\tilde{\Lambda}_2^{3N_- -2N}$					0	$\frac{2}{N_-}$	$-2M$
λ_2					0	$-\frac{2}{NN_-}$	0

Table 2: Quantum numbers of the dual $\mathrm{SU}(N) \times \mathrm{SU}(N-M)$ theory; we have written $N_- = N - M$ for concision.

(Note that the inversion of J is a sign of electric-magnetic duality, the generalization of $\tau \to -1/\tau$.) Matching of baryon numbers in the Seiberg duality assures that $(A)^{(N+M)} \sim (a)^{(N-M)}$. We will not attempt to match the $R_i^{(s)}$.

With these matchings, the dual theory has the global charges given in table 2. Remarkably, this theory has the same form as the previous one with $N \to N - M$. Thus the renormalization group flow is self-similar: the next step is that the $\mathrm{SU}(N)$ gauge group now becomes strongly coupled, and under a Seiberg duality transformation the full gauge group becomes $\mathrm{SU}(N-M)\times\mathrm{SU}(N-2M)$, and so forth.

This flow will stop, of course, at or before the point where $N - kM$ becomes zero or negative. Note that the Seiberg duality transformation is the same in both the so-called conformal window ($3N_c > N_f > \frac{3}{2}N_c$) and in the free magnetic phase ($3/2 \geq N_f > N_c + 1$). Even for $N_f = N_c + 1$ the effect on the superpotential described in [25] is not essential, since it is accounted for in the function F. The first significant changes occur when $N_f = N_c$, since for $N_f \leq N_c$ the classical moduli space is drastically modified. Thus, the RG flow just described proceeds step by step until the gauge group has the form $\mathrm{SU}(M + p) \times \mathrm{SU}(p)$, where $0 < p \leq M$. At this point we should do a more careful analysis, which we will carry out in the next section.

It is instructive to consider the type-IIA brane picture of the duality cascade of $\mathrm{SU}(N + M) \times \mathrm{SU}(N)$ theories.[8] To implement such theories we have to add M D4-branes stretched only one way between the D4-branes, rather than all around the circle [26, 23, 27, 30]. Such D4-branes are T-duals of the "fractional" branes which are the D5-branes wrapped over the 2-cycle of $\mathbb{T}^{11}$. These new D4-branes violate the balance of forces on the NS5-branes, and the latter undergo logarithmic

[8] We should remind the reader that the classical IIA picture of Seiberg duality, introduced by [26], has the feature that it is not a string duality but a motion which transforms one theory into its dual, one which a priori need not leave the infrared physics invariant. In particular, there is no direct relation between the classical brane motion, or semiclassical brane bending, and the actual dynamics of the field theories. To see that the IIA story gives the right answer, one should use its M-theory generalization [27, 28, 53, 54], but even there, the M5-brane, about which only holomorphic information is available, does not generally match the dynamics of the field theory, which is not holomorphic. These issues have been studied and explained in detail; see especially [29]. We mention this only to warn our readers not to take this paragraph for more than a heuristic argument.

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bending [27], which is the RG flow in this picture. Although the NS5 and NS5' brane bend along the circular x^6 direction, this does not force them to intersect, because they are oriented in perpendicular directions. However, their x^6 positions become equal somewhere away from the fractional branes, and it seems natural to interpret this as a divergence of the SU($N+M$) coupling. Under such circumstances it is natural to move the NS branes so as to eliminate this divergence, moving one of them once around the x^6 circle. When the NS- and NS'-brane cross during this motion, the $N+M$ fractional D4-branes shrink to zero size and then re-grow. In doing so, they flip their orientation and become anti-D4-branes. Meanwhile, the other N fractional branes stretch more than once around the circle, but where they are doubled they are partially cancelled by the $N+M$ anti-D4-branes. This leaves $N-M$ D4-branes in one segment and N in the other — exactly our starting point but with $N \to N-M$. After the crossing, the NS5-branes are still bent in the same directions as before, so again their x^6 positions become equal and we are led to repeat the motion around the circle. Finally, the number N becomes of order M and something more drastic should happen [31]. For this physics, the analysis of [28, 53, 54] becomes essential.

4. Chiral symmetry breaking and the deformation of the conifold

The solution of [12] is well behaved for large r but becomes singular at sufficiently small r. The solution must be modified in such a way that this singularity is removed. In this section we argue that the conifold (2.2) should be replaced by the deformed conifold

$$\sum_{i=1}^{4} z_i^2 = -2 \det_{i,j} z_{ij} = \epsilon^2 \,, \tag{4.1}$$

in which the singularity of the conifold is removed through the blowing-up of the S^3 of $\mathbb{T}^{11}$.

There are a number of arguments in favor of this idea. One suggestive observation is that in the solution of [12], the source of the singularity can be traced to the infinite energy in the F_3 field. At all radii there are M units of flux of F_3 piercing the S^3 of $\mathbb{T}^{11}$, and when the S^3 shrinks to zero size this causes F_3^2 to diverge. If instead the S^3 remained of finite size, as occurs in the deformed conifold, this problem would be evaded.

However, the most powerful argument that the conifold is deformed comes from the field theory analysis, which shows clearly that the spacetime geometry is modified by the strong dynamics of the infrared field theory. We will see that the theory has a deformed moduli space, with M independent branches, each of which has the shape of a deformed conifold. The branches are permuted by the $\mathbb{Z}_{2M}$ R-symmetry, which is spontaneously broken down to $\mathbb{Z}_2$. This breaking of the R-symmetry is exactly what we would expect in a pure SU(M) $\mathcal{N}=1$ Yang-Mills theory, although here it proceeds through scalar as well as gluino expectation values. The theory will also have domain walls, confinement, magnetic screening, and other related phenomena.

The complete analysis of the non-perturbative dynamics of the field theory in table 1 is mathematically intensive, and we have not attempted it. In this section we present a simplified version of the analysis which captures the physics which we are interested in. In an appendix we present more general (although still partial) results that show our conclusions are robust.

In particular, our goal is to discover what happens in the far infrared of the flow, where the D3-brane charge has cascaded (nearly) to zero and only the M fractional D3-branes remain. If there are no D3-branes left, we expect we have pure $\mathcal{N}=1$ Yang-Mills in the far infrared, a theory which breaks its $\mathbb{Z}_{2M}$ R-symmetry to $\mathbb{Z}_2$ and has M isolated vacua, domain walls, and confinement. However, while this may be correct, we have no access to the supergravity background through this analysis. What we need is a probe which can see if and how the fractional D3-branes have modified the conifold itself.

The right choice, it turns out, is to probe the space with a single additional D3-brane. In this case the gauge group is SU($M+1$)×"SU(1)" — in short, simply SU($M+1$) — with fields C_i and D_j in the $\mathbf{M+1}$ and $\overline{\mathbf{M+1}}$ representations, $i,j=1,2$, and with superpotential $W=\lambda C_i D_j C_k D_l \epsilon^{ik}\epsilon^{jl}$. Define $N_{ij}=C_i D_j$, which is gauge invariant. As in the discussion surrounding eq. (2.11), the expectation values of N_{ij} specify the position of the probe brane; in the classical theory, we have $\det_{i,j} N_{ij}=0$, indicating the probe is moving on the original, singular conifold. At low energy the theory can be written in terms of these invariants and develops the non-perturbative superpotential first written down by Affleck, Dine and Seiberg [32]

$$W_L=\lambda N_{ij}N_{k\ell}\epsilon^{ik}\epsilon^{j\ell}+(M-1)\left[\frac{2\Lambda^{3M+1}}{N_{ij}N_{k\ell}\epsilon^{ik}\epsilon^{j\ell}}\right]^{\frac{1}{M-1}}. \tag{4.2}$$

The equations for a supersymmetric vacuum are

$$0=\left(\lambda-\left[\frac{2\Lambda^{3M+1}}{(N_{ij}N_{k\ell}\epsilon^{ik}\epsilon^{j\ell})^M}\right]^{\frac{1}{M-1}}\right)N_{ij}\,. \tag{4.3}$$

The apparent solution $N_{ij}=0$ for all i,j actually gives infinity on the right-hand side. The only solutions are then

$$(N_{ij}N_{k\ell}\epsilon^{ik}\epsilon^{j\ell})^M=\frac{2\Lambda^{3M+1}}{\lambda^{M-1}}\,. \tag{4.4}$$

As predicted, this equation has M independent branches, in each of which $N_{ij}\,N_{k\ell}\,\epsilon^{ik}\,\epsilon^{j\ell}$ is a M^{th} root of $\Lambda^{3M+1}/\lambda^{M-1}$. The $\mathbb{Z}_{2M}$ discrete non-anomalous R-symmetry rotates $N_{ij}N_{k\ell}\epsilon^{ik}\epsilon^{j\ell}$ by a phase $e^{2\pi i/M}$, and thus the M branches transform into one another under the symmetry. In short, the $\mathbb{Z}_{2M}$ is spontaneously broken down to $\mathbb{Z}_2$. The low-energy effective superpotential is

$$W = M\lambda\langle N_{ij}N_{k\ell}\epsilon^{ik}\epsilon^{j\ell}\rangle \propto M\left[2\lambda\Lambda^{3M+1}\right]^{1/M} \tag{4.5}$$

which reflects the M branches. Most importantly, on each of these branches the classical condition on the N_{ij} has been modified to read

$$\det_{i,j} N_{ij} = \frac{1}{2}N_{ij}N_{k\ell}\epsilon^{ik}\epsilon^{j\ell} = \left(\frac{\Lambda^{3M+1}}{[2\lambda]^{M-1}}\right)^{1/M}. \tag{4.6}$$

Comparing with eq. (4.1) we see that the probe brane in the quantum theory moves on the deformed conifold; the classical singularity at the origin of the moduli space has been resolved through chiral symmetry breaking.

The above constraint on the expectation values for N_{ij} implies that in the perturbative region (where semiclassical analysis is valid) they can break the gauge group only down to SU(M), with no massless charged matter. This gauge theory is thus in the universality class of pure SU(M) Yang-Mills, and will share many of its qualitative properties. However, the existence of *massive* matter C_i, D_j in the fundamental representation of SU(M) (note that if $\langle N_{11}\rangle$ is large then C_2, D_2 have mass λN_{11}) implies that confinement occurs only in an intermediate range of distances. As in QCD with heavy quarks, pair production of the massive quarks breaks the confining flux tubes, so a linear potential between external sources exists only between the length scales $1/\sqrt{T}$ and m_q/T, where T is the string tension and m_q is the dynamical quark mass. For $\langle N_{11}\rangle \sim \langle N_{22}\rangle \sim (\Lambda^{3M+1}/\lambda^{M-1})^{1/2M}$, their minimal values, we expect little sign of a linear potential at any length scale, as in physical QCD. Only for $p=0$ do we expect confinement at all scales.

More generally, for $1 < p < M$, one obtains a moduli space corresponding to p probe branes moving on the deformed conifold. If $p \ll M$, both the SU(p) gauge coupling and its 't Hooft coupling are small at the strong-dynamics scale of SU($M+p$). Furthermore, the SU(p) factor has vanishing beta function in the far IR, where it has three adjoint chiral superfields (namely, three of the N_{ij}) and is essentially a copy of $\mathcal{N}=4$ Yang-Mills. Consequently, we expect no strong dynamics from the SU(p) sector, and the theory is very close to SU($M+p$) with $2p$ light flavors. In this case a similar analysis to the above is essentially correct. At large expectation values, the gauge theory is broken to SU(M) $\mathcal{N}=1$ Yang-Mills times SU(p) $\mathcal{N}=4$ Yang-Mills, with massive states in the bifundamental representation of the group factors. Details of this analysis are given in the appendix. As before, pair production of these massive states eliminates confinement at large distances; electric sources are screened by massive states which leave them charged only under the non-confining group SU(p).

The pattern of chiral symmetry breaking gives us another qualitative argument why the conifold must be deformed. The original conifold has a U(1)$_R$ symmetry under which the z_{ij} in (2.2) rotate by a phase. In table 1 we saw this was broken by instantons to $\mathbb{Z}_{2M}$, but for large M this is a $1/M$ effect and need not show up in the leading order supergravity. However, if we expect the infrared theory to behave similarly to pure $\mathcal{N}=1$ Yang-Mills, then we expect this symmetry to be spontaneously broken to $\mathbb{Z}_2$. This breaking is a leading-order effect and most definitely should be visible in the supergravity. The only natural modifications of the conifold are its resolution and its deformation; only the latter breaks the classical $\mathrm{U}(1)_R$ symmetry, and it indeed breaks it to $\mathbb{Z}_2$, as is obvious from eq. (4.1).

As a final argument, we consider expectations from the IIA/M-brane construction. Classically we have NS- and NS'-branes filling four-dimensional space and extending in the $v=x^4+ix^5$ and $w=x^8+ix^9$ directions respectively. They are separated along the compact direction x^6 by a distance a, which along with l_6 sets the two classical gauge couplings, as explained in (2.18). In one x^6 segment between the NS- and NS'-brane we suspend $M+1$ D4-branes; in the other there is only one D4-brane. A single complete wrapped D4-brane — our probe — is free to move anywhere in the v, w, x^7 space, independently of the other branes, while the other M suspended D4-branes are pinned to $v=w=0$. To understand the quantum theory, we must move to M-theory [28, 53, 54], where we combine x^6 with the new compact coordinate x^{10} using $t=e^{x^6+ix^{10}}$. Classically the equations for the NS- and NS'-brane are $w=0, t=1$ and $v=0, t=e^a$.

The M-theory expectation is that, in the quantum theory, the probe brane will become an independent M5-brane wrapped on the t directions, while the suspended D4-branes join with the NS- and NS'-branes to make a single M5-brane, which we will refer to as our MQCD brane. This type of behavior was first seen in $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetric Yang-Mills [27, 28, 53, 54]. Indeed the MQCD brane which appears in our case should be very similar to that of $\mathcal{N}=1$ super-Yang-Mills, since in the limit the x^6 direction becomes large they should become equal. The brane for super-Yang-Mills fills the coordinates x^0, x^1, x^2, x^3 and is embedded in the coordinates $v=x^4+ix^5, w=x^8+ix^9, t=e^{x^6+ix^{10}}$ as a Riemann surface defined through the equations

$$(vw)^M = \Lambda_L^{2M}\,, \qquad v^M = t\,. \tag{4.7}$$

Notice classically the equations include $vw=0$, corresponding to the presence of the NS- and NS'-brane. However, the quantum Yang-Mills M-brane has vw equal to a non-zero constant, and has M possible orientations, one for each possible phase of a condensate.

What is the connection with the deformed conifold? As shown in [33], a type-IIA NS-brane and NS'-brane satisfying the equation $vw=0$, that is, intersecting at a point, are T-dual to the conifold. This lifts without change to M-theory. We saw this equation appears in the construction of classical Yang-Mills, and it will appear in our classical theory as well. Meanwhile, if the NS- and NS'-branes are at the same t, that is, if they intersect, then they can be deformed into a single object with equation $vw = \text{constant} \neq 0$. This object is T-dual to the *deformed* conifold. Again this also lifts without change to M-theory. Now notice that the Yang-Mills M-brane has this as one of its defining eq. (4.7). This shows the NS- and NS'-brane have been glued together into a single object. Without the suspended D4-branes, this could only occur if the joined NS- and NS'-brane had equal t coordinates, but in the presence of the suspended D4-branes, which extend along the t direction, the NS- and NS'-branes can be separated in t, as in (4.7). Thus the Yang-Mills M-brane shows that the suspended D4-branes allow a quantum effect in M-theory by which the conifold can be deformed even when the two gauge couplings (2.18) are both finite.

In our case, we similarly expect the two Riemann surfaces — the probe and the MQCD brane — to have M branches, with a continuous variable specifying the position of the probe brane in the space, and a discrete variable labeling the orientation of the MQCD brane. However, when the probe is far away and the x^6 direction is large, our MQCD brane should closely resemble that of Yang-Mills. We therefore expect the equations governing it to have the same qualitative form. In particular, we expect that the $\mathbb{Z}_{2M}$ discrete symmetry rotating the phase of t by 2π is broken to $\mathbb{Z}_2$, through the modification of the equation $vw = 0$ to $(vw)^M = \text{constant}$. By T-duality this indicates that the classical conifold is quantum deformed by the fractional branes.

5. Back to supergravity: the deformed conifold ansatz

The field theory analysis of the previous section shows that the naive U(1) (really $\mathbb{Z}_{2M}$) R-symmetry is actually broken to a $\mathbb{Z}_2$. On the other hand, the SUGRA background (2.31) has an exact U(1) symmetry realized as shifts of the angular coordinate ψ on $\mathbb{T}^{11}$. The presence of this unwanted symmetry in the IR may also be the reason for the appearance of the naked singularity.

In this section we propose that the solution of this problem is to replace the conifold by its deformation (4.1) in the ansatz (2.31). This indeed breaks the U(1) symmetry $z_k \to e^{i\alpha} z_k$, $k = 1, \ldots, 4$, down to its $\mathbb{Z}_2$ subgroup $z_k \to -z_k$. Another reason to focus on the deformed conifold is that it gives the correct moduli space for the field theory, as shown in the previous section.

The metric of the deformed conifold was discussed in some detail in [14, 15, 34]. It is diagonal in the basis (2.4):

$$ds_6^2 = \frac{1}{2}\epsilon^{4/3} K(\tau) \left[\frac{1}{3K^3(\tau)} (d\tau^2 + (g^5)^2) + \cosh^2\left(\frac{\tau}{2}\right) \left[(g^3)^2 + (g^4)^2\right] + \right. \\ \left. + \sinh^2\left(\frac{\tau}{2}\right) \left[(g^1)^2 + (g^2)^2\right] \right], \tag{5.1}$$

where

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3} \sinh \tau} . \tag{5.2}$$

For large-τ we may introduce another radial coordinate r via

$$r^3 \sim \epsilon^2 e^\tau , \tag{5.3}$$

and in terms of this radial coordinate

$$ds_6^2 \longrightarrow dr^2 + r^2 ds_{\mathbb{T}^{11}}^2 . \tag{5.4}$$

The determinant of the metric (5.1) is

$$g_6 \sim \epsilon^8 \sinh^4 \tau , \tag{5.5}$$

which vanishes at $\tau = 0$. Indeed, at $\tau = 0$ the angular metric degenerates into

$$d\Omega_3^2 = \frac{1}{2}\epsilon^{4/3} \left(\frac{2}{3}\right)^{1/3} \left[\frac{1}{2}(g^5)^2 + (g^3)^2 + (g^4)^2\right] , \tag{5.6}$$

which is the metric of a round S^3 [14, 15]. The additional two directions, corresponding to the S^2 fibered over the S^3, shrink as

$$\frac{1}{8}\epsilon^{4/3} \left(\frac{2}{3}\right)^{1/3} \tau^2 \left[(g^1)^2 + (g^2)^2\right] . \tag{5.7}$$

In what follows we will set $\epsilon = 12^{1/4}$, so that $\frac{1}{2}\epsilon^{4/3}(2/3)^{1/3} = 1$.

The collapse of the S^2 implies that at $\tau = 0$ F_3 must lie within the remaining S^3,

$$F_3(\tau = 0) = M g^5 \wedge g^3 \wedge g^4 , \tag{5.8}$$

which may be shown to be a closed 3-form. On the other hand, for large τ, F_3 should approach its value

$$\frac{M}{2} g^5 \wedge \left(g^1 \wedge g^2 + g^3 \wedge g^4\right) \tag{5.9}$$

found in the UV ansatz (2.24). These two closed 3-forms differ by an exact one,

$$g^5 \wedge \left(g^1 \wedge g^2 - g^3 \wedge g^4\right) = d\left(g^1 \wedge g^3 + g^2 \wedge g^4\right) . \tag{5.10}$$

Therefore, the simplest ansatz which interpolates smoothly between $\tau = 0$ and large τ is

$$\begin{aligned} F_3 &= M \left\{g^5 \wedge g^3 \wedge g^4 + d[F(\tau)(g^1 \wedge g^3 + g^2 \wedge g^4)]\right\} \\ &= M\Big\{g^5 \wedge g^3 \wedge g^4(1 - F)) + g^5 \wedge g^1 \wedge g^2 F + \\ &\qquad + F' d\tau \wedge \left(g^1 \wedge g^3 + g^2 \wedge g^4\right) \Big\} , \end{aligned} \tag{5.11}$$

with $F(0) = 0$ and $F(\infty) = 1/2$. Note also that this ansatz preserves the $\mathbb{Z}_2$ symmetry which interchanges (θ_1, ϕ_1) with (θ_2, ϕ_2).

A similarly $\mathbb{Z}_2$-symmetric ansatz for B_2 is

$$B_2 = g_s M \Big[f(\tau) g^1 \wedge g^2 + k(\tau) g^3 \wedge g^4 \Big] . \tag{5.12}$$

Using the identity

$$g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) = -d(g^1 \wedge g^2 - g^3 \wedge g^4) , \tag{5.13}$$

we find that

$$\begin{aligned} H_3 = dB_2 = g_s M \Big[& d\tau \wedge (f' g^1 \wedge g^2 + k' g^3 \wedge g^4) + \\ & + \frac{1}{2}(k - f) g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \Big] . \end{aligned} \tag{5.14}$$

We further have

$$\mathcal{F}_5 = B_2 \wedge F_3 = g_s M^2 \ell(\tau) g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 , \tag{5.15}$$

where

$$\ell = f(1 - F) + kF . \tag{5.16}$$

The most general radial ansatz for the 10-d metric, consistent with the symmetries of the deformed conifold, is

$$\begin{aligned} ds^2_{10} = & A^2(\tau) dx_n dx_n + B^2(\tau)(d\tau^2) + C^2(\tau)(g^5)^2 + \\ & + D^2(\tau) \left[(g^3)^2 + (g^4)^2 \right] + E^2(\tau) \left[(g^1)^2 + (g^2)^2 \right] . \end{aligned} \tag{5.17}$$

The reason we are allowed to assume that $A, \ldots, E$ depend only on τ is that before the introduction of the 3-form fields, the metric has the form (5.17), and our ansatz for F_3 and H_3 does not break this symmetry. The flux of F_3 is distributed uniformly over the S^3 near the apex of the deformed conifold; therefore, the M D5-branes wrapped over the S^2 may be thought of as smeared over the S^3.

It is not hard to check that $F_{3\mu\nu\lambda} H_3^{\mu\nu\lambda} = 0$, which implies that the RR scalar vanishes. It is not a priori clear whether the dilaton is constant for the deformed solution, but in what follows we will assume that such a background does exist, i.e. that

$$g_s^2 F_3^2 = H_3^2 . \tag{5.18}$$

Furthermore, guided by the simple form of the solution constructed in [12] and reviewed in section 2, we will assume that the 10-d metric takes the following form:

$$ds^2_{10} = h^{-1/2}(\tau) dx_n dx_n + h^{1/2}(\tau) ds^2_6 , \tag{5.19}$$

where ds^2_6 is the metric of the deformed conifold (5.1). This is the same type of "D-brane" ansatz as (2.31), but with the conifold replaced by the deformed conifold as the transverse space. This form will also permit additional D3-brane probes to be directly included in the ansatz.

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The type-IIB equations satisfied by the 3-form fields are

$$d(e^{\phi} \star F_3) = F_5 \wedge H_3 \,, \qquad d(e^{-\phi} \star H_3) = -g_s^2 F_5 \wedge F_3 \,. \tag{5.20}$$

First, let us calculate

$$\star \mathcal{F}_5 \sim g_s M^2 dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\tau \frac{\ell(\tau)}{K^2 h^2 \sinh^2(\tau)} \,. \tag{5.21}$$

To write down the first equation we need

$$\begin{aligned} \star F_3 = &\, M h^{-1} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \\ &\wedge \Big[(1-F) \tanh^2\left(\frac{\tau}{2}\right) d\tau \wedge g^1 \wedge g^2 + \\ &\quad + F \coth^2\left(\frac{\tau}{2}\right) d\tau \wedge g^3 \wedge g^4 + F' g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \Big] \,. \end{aligned} \tag{5.22}$$

Assuming a constant ϕ and using (5.13) we find

$$(1-F)\tanh^2\left(\frac{\tau}{2}\right) - F \coth^2\left(\frac{\tau}{2}\right) + 2h \frac{d}{d\tau}(h^{-1}F') = \alpha (k-f) \frac{\ell}{K^2 h \sinh^2 \tau} \,, \tag{5.23}$$

where α is a normalization factor proportional to $(g_s M)^2$.

Let us turn to the second of the eqs. (5.20). Since

$$\begin{aligned} \star H_3 = &\, -g_s M h^{-1} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \\ &\wedge \Big[g^5 \wedge \left(k' \tanh^2\left(\frac{\tau}{2}\right) g^1 \wedge g^2 + f' \coth^2\left(\frac{\tau}{2}\right) g^3 \wedge g^4 \right) - \\ &\quad - \frac{1}{2}(f-k) d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \Big] \,, \end{aligned} \tag{5.24}$$

we find

$$\begin{aligned} h \frac{d}{d\tau}\left(h^{-1} \coth^2\left(\frac{\tau}{2}\right) f' \right) - \frac{1}{2}(f-k) &= \alpha \frac{\ell(1-F)}{K^2 h \sinh^2 \tau} \,, \\ h \frac{d}{d\tau}\left(h^{-1} \tanh^2\left(\frac{\tau}{2}\right) k' \right) + \frac{1}{2}(f-k) &= \alpha \frac{\ell F}{K^2 h \sinh^2 \tau} \,, \end{aligned} \tag{5.25}$$

where α is the same normalization factor as in (5.23).

We have been assuming that the dilaton is constant. The equation that guarantees this is (5.18). Writing it out with our ansatz gives

$$\frac{(k')^2}{\cosh^4(\tau/2)} + \frac{(f')^2}{\sinh^4(\tau/2)} + \frac{2(f-k)^2}{\sinh^2 \tau} = \frac{(1-F)^2}{\cosh^4(\tau/2)} + \frac{F^2}{\sinh^4(\tau/2)} + \frac{8(F')^2}{\sinh^2 \tau} \,. \tag{5.26}$$

In order to complete the system of equations we need the Einstein equations for the metric. In view of the simplified ansatz (5.19) for the metric it is sufficient to use the trace of the Einstein equation:

$$h^{-3/2} \nabla_6^2 h \sim g_s^2 F_3^2 + H_3^2 = 2 g_s^2 F_3^2 \,, \tag{5.27}$$

where now ∇_6^2 is the laplacian on the deformed conifold. Using (5.1), we find that the explicit form of this equation is

$$\frac{1}{\sinh^2\tau}\frac{d}{d\tau}(h'K^2(\tau)\sinh^2\tau) = -\frac{\alpha}{4}\left[\frac{(1-F)^2}{\cosh^4(\tau/2)} + \frac{F^2}{\sinh^4(\tau/2)} + \frac{8(F')^2}{\sinh^2\tau}\right]. \quad (5.28)$$

5.1 The first-order equations and their solution

In searching for BPS saturated supergravity backgrounds, the second-order equations should be replaced by a system of first-order ones (see, for instance, [35, 36]). Luckily, this is possible for our ansatz. We have been able to find a system of simple first-order equations, from which (5.23), (5.25), (5.26) and (5.27) follow:

$$f' = (1-F)\tanh^2\left(\frac{\tau}{2}\right), \qquad k' = F\coth^2\left(\frac{\tau}{2}\right), \qquad F' = \frac{1}{2}(k-f), \quad (5.29)$$

and

$$h' = -\alpha\frac{f(1-F)+kF}{K^2(\tau)\sinh^2\tau}. \quad (5.30)$$

Note that the first three of these equations, (5.29), form a closed system and need to be solved first. In fact, these equations imply the self-duality of the complex 3-form with respect to the metric of the deformed conifold: $\star_6 G_3 = iG_3$. Inspection of these equations shows that the small τ behavior is[9]

$$f \sim \tau^3, \qquad k \sim \tau, \qquad F \sim \tau^2. \quad (5.31)$$

On the other hand, for large τ the 3-forms have to match onto the conifold solution [12],

$$f \longrightarrow \frac{\tau}{2}, \qquad k \longrightarrow \frac{\tau}{2}, \qquad F \longrightarrow \frac{1}{2}. \quad (5.32)$$

Remarkably, it is possible to find the solution with these boundary conditions in closed form. Combining (5.29) we find the following second-order equation for F:

$$F'' = \frac{1}{2}\left[F\coth^2\left(\frac{\tau}{2}\right) + (F-1)\tanh^2\left(\frac{\tau}{2}\right)\right]. \quad (5.33)$$

The solution is

$$F(\tau) = \frac{\sinh\tau - \tau}{2\sinh\tau}, \quad (5.34)$$

from which we obtain

$$\begin{aligned} f(\tau) &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1), \\ k(\tau) &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1). \end{aligned} \quad (5.35)$$

[9]It is also possible to shift f and k by the same constant. The effect of this shift will be considered in section 5.2.

Now that we have solved for the 3-forms on the deformed conifold, the warp factor may be determined by integrating (5.30). First we note that

$$\ell(\tau) = f(1-F) + kF = \frac{\tau \coth\tau - 1}{4\sinh^2\tau}(\sinh 2\tau - 2\tau)\,. \tag{5.36}$$

This behaves as τ^3 for small τ. It follows that, for small τ,

$$h = a_0 + a_1\tau^2 + \cdots\,. \tag{5.37}$$

For large τ we impose, as usual, the boundary condition that h vanishes. The resulting integral expression for h is

$$h(\tau) = \alpha\frac{2^{2/3}}{4}\int_\tau^\infty dx \frac{x\coth x - 1}{\sinh^2 x}(\sinh(2x) - 2x)^{1/3}\,. \tag{5.38}$$

We have not succeeded in evaluating this in terms of elementary or well-known special functions. For our purposes it is enough to show that

$$h(\tau \to 0) \longrightarrow a_0\,; \qquad h(\tau\to\infty) \longrightarrow \frac{3}{4}2^{1/3}\alpha\tau e^{-4\tau/3}\,. \tag{5.39}$$

This is non singular at the tip of the deformed conifold and, from (5.3), matches the form of the large-τ solution (2.38). The small τ behavior follows from the convergence of the integral (5.38), while at large τ the integrand becomes $\sim xe^{-4x/3}$.

Thus, for small τ the ten-dimensional geometry is approximately $\mathbb{R}^{3,1}$ times the deformed conifold:

$$ds_{10}^2 \longrightarrow a_0^{-1/2}dx_n dx_n + a_0^{1/2}\left(\frac{1}{2}d\tau^2 + d\Omega_3^2 + \frac{1}{4}\tau^2\left[(g^1)^2 + (g^2)^2\right]\right)\,. \tag{5.40}$$

Very importantly, for large $g_s M$ the curvatures found in our solution are small everywhere. This is true even far in the IR. Indeed, since the integral (5.38) converges,

$$a_0 \sim \alpha \sim (g_s M)^2\,. \tag{5.41}$$

Therefore, the radius-squared of the S^3 at $\tau = 0$ is of order $g_s M$, which is the 't Hooft coupling of the gauge theory found far in the IR. As long as this is large, the curvatures are small and the SUGRA approximation is reliable.

We have now seen that the deformation of the conifold allows the solution to be non singular. Qualitatively, this is because the conserved F_3 flux prevents the 3-cycle from collapsing. This is why we expect to find a metric with a collapsing 2-cycle but finite 3-cycle, and these are the properties of the deformed conifold.

It may be of further interest to consider more general metrics of the form (5.17), and to allow the dilaton to vary. In that event it still seems likely that the qualitative properties of the solution near the apex will not change.

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5.2 Correspondence with the gauge theory

In this section we point out some interesting features of the SUGRA background we have found and show how they realize the expected phenomena in the dual field theory. In particular, we will now demonstrate that there is confinement and magnetic screening, and argue that there are domain walls and baryon vertices with a definite mass scale. In many ways our results resemble those found in the $\mathcal{N} = 1^*$ theory [37], but the specific details are quite different; the confining vacua of $\mathcal{N} = 1^*$ involve a spacetime with a spherical 5-brane sitting in it, while our present spacetime is purely given by supergravity.

First we should ask the question: how does the dimensional transmutation manifest itself in supergravity? The answer is related to the presence of parameter ϵ in the deformed conifold metric (5.1). Reinstating this parameter is accomplished through

$$ds_6^2 \longrightarrow \epsilon^{4/3} ds_6^2\,. \tag{5.42}$$

We are then free to redefine $h \to h\epsilon^{-8/3}$ to remove the ϵ dependence from the transverse part of the metric. Very importantly, the dependence then appears in the longitudinal part, and the metric assumes the form

$$ds_{10}^2 = h^{-1/2}(\tau) m^2 dx_n dx_n + h^{1/2}(\tau) ds_6^2\,, \tag{5.43}$$

so that $m \sim \epsilon^{2/3}$ sets the 4-d mass scale. This scale then appears in all 4-d dimensionful quantities.

Now let us see the theory has confining flux tubes. The key point is that in the metric for small τ (5.40) the function multiplying $dx_n dx_n$ approaches a constant. This should be contrasted with the AdS_5 metric where this function vanishes at the horizon, or with the singular metric of [12] where it blows up. Consider a Wilson contour positioned at fixed τ, and calculate the expectation value of the Wilson loop using the prescription [39, 40]. The minimal area surface bounded by the contour bends towards smaller τ. If the contour has a very large area A, then most of the minimal surface will drift down into the region near $\tau = 0$. From the fact that the coefficient of $dx_n dx_n$ is finite at $\tau = 0$, we find that a fundamental string with this surface will have a finite tension, and so the resulting Wilson loop satisfies the area law. Since for large $g_s M$ the SUGRA description is reliable for all τ, we seem to have found a "pure supergravity proof" of confinement in $\mathcal{N} = 1$ gauge theory. A similar result was found previously in [37] but involved a spacetime containing an NS5-brane with D3-brane charge. A simple estimate shows that the string tension scales as

$$T_s \sim \frac{m^2}{g_s M}\,. \tag{5.44}$$

To see that magnetic charge is screened, we must identify the correct massive magnetically-charged source. The correct choice is a fractional D1-brane, that is, a D3-brane wrapped on the S^2 of $\mathbb{T}^{11}$, attached to the boundary of the space at $\tau = \infty$. On the six-dimensional deformed conifold the S^2 is fibered over τ such that the resulting three-dimensional bundle has only one boundary, at $\tau = \infty$; near $\tau = 0$ the S^2 shrinks to zero size and the bundle locally has topology $\mathbb{R}^3$. Therefore, a D3-brane with a single boundary can be wrapped on this bundle, corresponding to a fractional D1-brane attached at $\tau = \infty$ which quietly ends at $\tau = 0$. Strictly speaking, this only shows monopole charge is not confined; to show it is screened one must go further and show this object does not couple to any massless modes.

As we showed in section 2, the field theory has an anomaly-free $\mathbb{Z}_{2M}$ R-symmetry at all scales. The UV limit of our background, which coincides with the solution found in [12], has a U(1) R-symmetry associated with the rotations of the angular coordinate ψ. For large M it is is somewhat difficult to distinguish between the U(1) and its discrete subgroup $\mathbb{Z}_{2M}$. In fact, the anomaly in the U(1), which breaks it down to $\mathbb{Z}_{2M}$, is an effect of fractional D-instantons, the euclidean D-string world sheets propagating inside $\mathbb{T}^{11}$. The Wess-Zumino term present in the D-string action, which is associated with the topologically non-trivial F_3, has to be quantized (this is simply the F_3 flux quantization). As a result, the phase in the D-string path integral assumes $\mathbb{Z}_{2M}$ rather than U(1) values.

Our metric provides a geometrical realization for the phenomenon of chiral symmetry breaking found in the field theory; the dynamical breaking of the $\mathbb{Z}_{2M}$ down to $\mathbb{Z}_2$ occurs via the deformation of the conifold. In the pure supergravity limit we have discussed, the spontaneous chiral symmetry breaking generates an η'-like Goldstone boson (the zero mode in our solution corresponding to rotation of the coordinate ψ), which must get a mass of order $1/M$ from these fractional instantons. To see how this mass arises, and how it relates to the domain walls which we discuss in a moment, would be very interesting.

It is by now clear why the conifold ansatz adopted in [12] and reviewed in section 2 is too restrictive: it has the U(1) symmetry everywhere. On the other hand, our deformed conifold ansatz breaks it down to $\mathbb{Z}_2$, with the U(1) symmetry becoming asymptotically restored at large radius. Thus, the deformation of the conifold ties together several crucial IR effects: resolution of the naked singularity found in [12], breaking of the chiral symmetry down to $\mathbb{Z}_2$, and quark confinement. At the same time, the deformation does not destroy the logarithmic running of the couplings found in [12] because it does not affect the geometry far in the UV.

Due to the deformation, the full SUGRA background has a finite 3-cycle. We now interpret various branes wrapped over this 3-cycle in terms of the gauge theory. Note that the 3-cycle has the minimal volume near $\tau = 0$, hence all the wrapped branes will be localized there. This should be contrasted with wrapped branes in $AdS_5 \times X_5$ where they are allowed to have an arbitrary radial coordinate. A wrapped D3-

brane plays the role of a baryon vertex which ties together M fundamental strings. Note that for $M = 0$ the D3-brane wrapped on the S^3 gave a dibaryon [10]; the connection between these two objects becomes clearer when one notes that for $M > 0$ the dibaryon has M uncontracted indices, and therefore joins M external charges. Meanwhile, a D5-brane wrapped over the S^3 appears to play the role of the domain wall separating two inequivalent vacua of the gauge theory. As we expect, flux tubes can end on this object [38], and baryons can dissolve in it; as in [37], we may also build the domain walls from the baryons. Indeed, D3- and D5-branes play the roles of baryon vertices and domain walls in $\mathcal{N} = 1^*$; however in that case they do not wrap a cycle but instead have a boundary on the NS5-brane in the space [37]. Calculations using the metric (5.43) show that the baryon mass is

$$M_b \sim mM \,, \tag{5.45}$$

while the D5-brane domain wall tension is

$$T_{\rm wall} \sim \frac{1}{g_s} m^3 \,. \tag{5.46}$$

Additionally, one can obtain the glueball spectrum in this theory. To do so requires finding the spectrum of eigenmodes of various supergravity fields in the metric background we have constructed. Since the background is known explicitly as a function of τ, the calculation should be no more difficult than in [41, 42]. Unlike the case of $\mathcal{N} = 1^*$, where the presence of a narrow throat near a single NS5-brane could make the computation potentially unreliable for the lowest modes [37], there is no possible subtlety here, as the bulk space is large and everywhere non singular. Of course, there will be Kaluza-Klein modes on the S^3 which are not present in the pure $\mathcal{N} = 1$ Yang-Mills theory. These are analogous to the extra modes which appear in both [43] and [37]; their presence is expected, since they are necessary whenever pure $\mathcal{N} = 1$ Yang-Mills is embedded into a theory that is fully in the supergravity regime. Only in the limit of pure $\mathcal{N} = 1$ Yang-Mills, which we discuss below, can they be removed. A simple estimate of the glueball and KK modes masses shows that, in the SUGRA limit both scale as $m/(g_s M)$. Comparing with the string tension, we see that

$$T_s \sim g_s M (m_{\rm glueball})^2 \,. \tag{5.47}$$

Thus, there is a large separation of scales between string tension and glueball mass in supergravity (a similar problem was observed in [41, 42]) which goes away at small $g_s M$.

Finally, we should address the possibility that N is not a multiple of M. Note that in our solution the 5-form flux vanishes for $\tau = 0$:

$$\int F_5 = \ell(\tau) \sim \tau^3 \,. \tag{5.48}$$

This suggests that the IR solution given above describes a large number M of wrapped D5-branes without any D3-branes. Therefore, for small τ the background should be dual to SU(M) gauge theory (the SUGRA is reliable only if both M and g_sM are large). More generally, however, the field theory analysis tells us that theories that may arise in the IR have gauge groups $\mathrm{SU}(M+p)\times\mathrm{SU}(p)$, with $M > p \geq 0$. If M is large and p is of order 1, then the dual supergravity background should be the same as for $p = 0$, to leading order in M. The extra p colors should come from p actual D3-branes, placed at various points in our background. Then the moduli space for each D3-brane is essentially the deformed conifold, in agreement with the field theory analysis. When far from $\tau = 0$, the D3-branes represent the IR $\mathcal{N} = 4$ SU(p) factor in the theory. The 't Hooft coupling on these branes is $g_sp \ll 1$, so when they are brought to $\tau = 0$ the theory represented is essentially $\mathrm{SU}(M+p)$ with $2p$ classically massless flavors and a quartic superpotential. The non-perturbative analysis of this theory, given in section 4 and in the appendix, then applies, giving chiral symmetry breaking and a moduli space with M branches.

Note that confinement is lost in the presence of the D3-branes, in agreement with the field theory. Strings hanging from the boundary can simply end on the D3-branes, corresponding to the statement that external sources are screened by massive dynamical quarks and end up carrying only SU(p) charge.[10] The corresponding Wilson loop will have a perimeter law. Of course if the quarks are heavy (i.e. if the D3-branes are at large τ) then relatively short flux tubes should be stable. It would be interesting to actually demonstrate this fact, which follows not from topology but from quantum dynamics.

On the other hand, if p is of the same order as M, then the flux due to the D3-branes is large and should be included in the SUGRA solution. First, let us try to change the boundary condition on F_5 so that F_5 no longer vanishes at $\tau = 0$ but is $\sim p$. We find a consistent solution for the 5-form by replacing $\ell(\tau) \to \ell(\tau) + C$, where C is a constant of order $p/(g_sM^2)$. From (5.30) we find that the effect of this on the warp factor is $h \to h + \tilde{h}$ where

$$\tilde{h}(\tau) = \alpha C \int_\tau^\infty dx \frac{1}{K^2(x)\sinh^2 x} \,. \tag{5.49}$$

This yields a singular behavior of $\tilde{h}$ for small τ:

$$\tilde{h} \sim \frac{\alpha C}{\tau} \,. \tag{5.50}$$

The new behavior of h does change significantly the physical interpretation of the solution. Now the coefficient of the $dx_n dx_n$ term scales as $\tau^{1/2}$ for small τ;

[10] Similar findings were also obtained in $\mathcal{N} = 1^*$ [37]. Many of the $\mathcal{N} = 1^*$ vacua have dynamical massive W-bosons, whose pair production eliminates confinement. The representation of this gauge theory physics in the string theory is closely related to the representation presented here and in the last paragraph of this section.

hence, the Wilson loop no longer satisfies the area law. Again, we find agreement with the field theory. This gravity background corresponds to making the charged matter as light as possible (that is, making the expectation values of the scalar fields all as small as possible). In this regime we expect no metastable flux tubes; the dynamical charges in the fundamental representation of $\mathrm{SU}(M+p)$ will screen external electric sources, until the sources are charged only under $\mathrm{SU}(p)$, which does not confine.

Thus, the new behavior of the metric (5.50) incorporates the loss of confinement found upon addition of dynamical quarks. However, supergravity may receive large corrections in the small τ region because curvatures blow up at $\tau = 0$ where we find a singular horizon.[11] Thus, requiring that F_5 does not vanish at $\tau = 0$ actually causes a singularity. Can we construct a non-singular SUGRA solution which incorporates screening? We believe that the correct approach may be to add D3-brane sources with total charge p (this way F_5 may smoothly turn on from zero at $\tau = 0$ to p at some finite value of τ). This idea also agrees with the incorporation of small p via actual D3-branes. We postpone construction of such non-singular "Coulomb branch" solutions until a later publication.

5.3 The dual of pure $\mathcal{N} = 1$ Yang-Mills theory

As we have shown above, supergravity serves as a reliable dual of a cascading $\mathrm{SU}(N+M) \times \mathrm{SU}(M)$ gauge theory, provided that g_sM is very large. We have also shown that, under appropriate circumstances, at the bottom of the cascade, we essentially have an $\mathrm{SU}(M)$ theory, with the other gauge group disappearing. An immediate question that arises is: can our results be used to learn something about the pure glue $\mathcal{N} = 1$ theory?

To start answering this question, let us note that the field B_2 is multiplied by g_sM, while the jumps in the cascade occur after B_2 has changed by an amount of order 1. Thus, the range of τ which describes any particular gauge group in the cascade is of order $1/(g_sM)$. This implies the supergravity regime is not sufficient for constructing such a dual, because for large g_sM the cascade jumps occur very frequently, and we find the pure glue theory only for small τ. There, at the tip of the deformed conifold, both B_2 and F_5 are very small, F_3 is of order M, and the metric is approximately given by (5.40).

To have a reliable dual of the pure glue theory, valid for a large range of τ, we need to take the limit of *small* g_sM (and thus small B_2, holding M fixed) which is the opposite of the limit where supergravity has small corrections. In this limit the S_3 at the apex of the conifold becomes small and the space acquires large curvature. This situation is familiar from previous studies aimed at finding a string theory dual of a pure glue gauge theory [43, 37].

[11] We are grateful to A. Tseytlin for useful discussions of this point.

Nevertheless, our work does constitute progress towards formulating a stringy dual because our SUGRA background captures the correct topology of the resulting string background. Indeed we are led to conjecture that the type-IIB string dual of the pure glue $\mathcal{N} = 1$ SU(M) theory is given by a $g_s M \to 0$ limit of a warped deformed conifold background, with M units of the F_3 flux piercing its 3-cycle, and with B_2 and F_5 approaching zero at the apex. This would be the space generated by the fractional D3-branes alone, with *no* admixture of regular D3-branes. Hence it is relevant to pure SU(M) theory with no quark flavors. Of course, studying such a theory for small $g_s M$ is difficult due to the well-known problems with RR flux and large curvature. However, the self-dual 5-form flux, which brings in some additional problems, is small, which raises hopes of a novel sigma model formulation.

We note also that the addition of a small number of D3-branes to this story will permit the study of the SU$(M+p) \times$ SU(p) theory, which essentially reduces, for small g_s and $p \ll M$, to SU$(M+p)$ with $2p$ flavors and an all-important quartic superpotential. It is far from certain that this construction can give any insight into QCD, since the light charged scalars play such a central role in the dynamics. However, if these scalars can easily be removed (along with the gauginos) through explicit supersymmetry breaking, there might be additional interest in this approach.

6. Discussion

We have not addressed the question of how to compute field theory correlation functions in this context, where our space does not approach Anti-de-Sitter space at large r. However, it is easy to see this space still has a boundary, and from the behavior of $h(\tau)$ it is clear that the logarithm is a subleading effect at large r. Correspondingly, at large $N \gg M$, there is a sense in which the operators in the field theory have the same spectrum that they have for $M = 0$, since $\gamma \approx -1/2$. We therefore believe that for low-lying supergravity modes, corresponding to operators of dimension much less than M, the story will not be modified in a significant way from that discussed in [2, 3]. For operators of dimension $\Delta > \frac{3}{2}M$, we expect more interesting effects. These operators appear to exist at scales where $\frac{3}{2}N > \Delta$, but should be eliminated when $\frac{3}{2}N < \Delta$. In the gauge theory, it is known what should occur [44]; operators of high dimension present classically are actually removed by quantum effects, which in the low-energy dual theory appear as simple group theory. On the gravity side we may speculate that high-lying bulk modes propagate in from the boundary until the region where $N \sim \frac{3}{2}\Delta$; there $\mathbb{T}^{11}$ has shrunk down such that these modes blow up into the "giant gravitons" of [45]. Only modes with dimension $\Delta < \frac{3}{2}M$ can propagate all the way to $\tau = 0$.

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It is easy to see that our story of the duality cascade can be orientifolded. This is obvious from the type-IIA string theory brane construction. It is also clear from the corresponding SO $\times$ Sp gauge theory, although we have not analyzed the field theory dynamics to see how the orientifolded conifold is deformed. A number of other modifications, including theories whose IIA version involves multiple NS- and NS'-branes, could potentially be interesting. This might be especially true for theories which are qualitatively different in the infrared from pure Yang-Mills, such as those studied in [46].

Another interesting choice would be to orbifold the theory along the lines of [37], so that the low energy theory is non-supersymmetric SU$(M)^2$ with a Dirac fermion in the bifundamental representation. In contrast to the case studied in [37], the masslessness of this fermion would be exact, as it is guaranteed by the $\mathbb{Z}_{2M}$ R-symmetry, and therefore chiral symmetry breaking and confinement in this QCD-like theory could be exhibited in the supergravity regime.

Finally, it is interesting to resurrect a scenario discarded five years ago for its apparent absurdity. Namely, it is conceivable that the standard model — a small gauge group — itself lies at the base of a duality cascade. This is certainly possible, since the addition of supersymmetry and some appropriately chosen massive matter at the TeV scale easily could make the theory into one which could emerge from such an RG flow. In [47] it was in fact pointed out that this was the natural scenario if the standard model, with its very small gauge groups, is a low-energy Seiberg-dual description of some other theory; every natural choice for an ultraviolet theory has a larger gauge group than the standard model, and typically hits a Landau pole below the Planck scale, requiring additional duality transformations, still larger gauge groups, more Landau poles, and continuation *ad nauseum*. This was termed the "duality wall" (since in some cases the duality transformations piled up so fast that an infinite number were required in a finite energy range). But now we see this continuous generation of larger and larger gauge groups — ugly and unmotivated within field theory, and driving the field theory into highly non-perturbative regimes — can correspond to a perfectly natural spacetime background on which strings may propagate. If we imagine that the ultraviolet of the duality cascade is cut off in a compact space (along the lines of [48], following [49]) we may conjecture that the standard model coupled to gravity is best described, at high energy, by a compactified string theory on a space with a logarithmic (or otherwise warped) throat, with the weakly coupled standard model emerging as a good description only at energies below, say, 1–100 TeV. Such a model provides another possible way, somewhat related to ideas of [48]–[52], to explain the hierarchy between the gravitational and electroweak scales: it is perhaps given by TeV $= m_{\rm Pl} \times e^{-cN/M}$, where M is of order 2 to 5, c is a number of order one, and N is the number of colors of the gauge group at the Planck scale.

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A. Analysis of the field theory

In this appendix we analyze the field theory in somewhat greater detail, confirming and extending the results of section 3.

First, we may check the results of section 3 in another region of moduli space. Consider first SU$(M+1)$ with two flavors. Suppose we permit C_1 and D_1 to have equal expectation values v, so that $N_{11} = v^2$. This breaks the SU$(M+1)$ to SU(M). If λ were zero, this would leave SU(M) with one flavor C_2 and D_2, plus two gauge singlets $\langle C_1 \rangle D_2 = N_{12}$ and $C_2 \langle D_1 \rangle = N_{21}$; the corresponding strong coupling scale would be Λ^{M+1}/v^2. However, the presence of non-zero λ gives mass to these fields, leaving the SU(M) gauge theory with a flavor of mass λv^2. The effective lagrangian is then

$$W = 2\lambda v^2 N_{22} + \left[\frac{\Lambda^{M+1}/v^2}{N_{22}}\right]^{\frac{1}{M-1}} \tag{A.1}$$

which again leads to M branches with the correct values of $N_{ij}N_{k\ell}\epsilon^{ik}\epsilon^{j\ell}$.

That our discussion of the SU$(M+1)$ theory in section 3 was only part of the story can be seen by starting one step higher, with SU$(2M+1) \times$ SU$(M+1)$, which reduces after one duality transformation to the SU$(M+1)$ case. The SU$(2M+1)$ gauge group has one more flavor than color, and therefore, as $\lambda_1 \to 0$, the theory is governed by the results of [25]. For $\lambda_1 = 0$ the superpotential must go over to

$$W \longrightarrow \frac{\det P^a_{ijb}}{\Lambda_1^{4M+1}} - C_{ia} P^a_{ijb} D^b_j\,, \tag{A.2}$$

where a, b are color indices of SU$(M+1)$, and $P \sim AB$, $C \sim A^{2M+1}$, $D \sim B^{2M+1}$. From this we learn the function $F_1(I_1, J_1, R_1^{(1)})$ is not equal to one, and in fact, in the limit $\lambda_1 \to 0$, that is, $I_1, J_1 \to 0$, we have $F_1(I_1, J_1, R_1^{(1)}) \to \sqrt{I_1/J_1} f(R_1^{(1)})$. The low energy theory is then SU$(M+1)$ with two flavors C_i, D_i but with superpotential

$$W = \lambda_2 C_i D_j C_k D_l \epsilon^{ik}\epsilon^{jl} F_2(I_2, J_2)\,. \tag{A.3}$$

Here $(C_i D_j C_k D_l \epsilon^{ik}\epsilon^{jl})$ is the only invariant involving C and D; there are no R_2 ratios. The low energy effective superpotential is now

$$W_L = \lambda N_{ij} N_{k\ell} \epsilon^{ik}\epsilon^{j\ell} F_2(I_2, J_2)\,. \tag{A.4}$$

Note that $F_2(I_2 \to 0, J_2 \to 0) = 1 + \sqrt{I_2/J_2}$; some other limits can be studied but will not be needed here. The vacuum equations are

$$0 = \lambda \left[F(I_2, J_2) + I_2 \frac{\partial F(I_2, J_2)}{\partial I_2} \right] N_{ij} \,. \tag{A.5}$$

This gives an equation for $I_2 \propto (N_{ij} N_{k\ell} \epsilon^{ik} \epsilon^{j\ell})^{2M}$, whose solution must be

$$I_2 = G(J_2) \,. \tag{A.6}$$

The holomorphic function $G(J_2)$ is not zero everywhere (since for $I_2 \to 0$, $J_2 \to 0$ it is not zero) so it can only be zero at special points. Consequently $N_{ij} N_{k\ell} \epsilon^{ik} \epsilon^{j\ell}$ is generally non zero. Since the $\mathbb{Z}_{2M}$ symmetry rotates $N_{ij} N_{k\ell} \epsilon^{ik} \epsilon^{j\ell}$ by $e^{2\pi i/M}$, we again find M separate branches. Again there are no restrictions on the individual values of the N_{ij}, so each branch takes the form of a deformed conifold, with a non-zero superpotential. Thus we obtain the same result as before; only the magnitude of the deformation is modified from our previous analysis.

This analysis is too weak to rule out the possibility that there might be several independent solutions for I_2 given a single value of J_2. This would lead to several sets of branches, each set consisting of M copies of a deformed conifold; the different sets would have deformations of different magnitudes. In the limit $\Lambda_1 \to \infty$ only one set would remain, as in our earlier analysis of the SU$(M+1)$ theory.

Next, we consider the case of SU$(M+p) \times$ SU(p), $0 < p < M$. We will first perform the analysis by taking the SU(p) coupling small. We define $(N_{ij})^\alpha_\beta = (C_i)^\alpha_a (D_i)^a_\beta$, where α, β are SU(p) indices and a, b are SU$(M+p)$ indices. If the SU(p) coupling were set to zero, then we would have an SU$(M+p)$ gauge theory with $2p$ flavors. An Affleck-Dine-Seiberg superpotential would be generated, giving

$$W = \lambda (N_{ij})^\alpha_\beta (N_{k\ell})^\beta_\alpha \epsilon^{ik} \epsilon^{j\ell} + (M-p) \left(\frac{\Lambda_1^{3M+p}}{\det_{ij,\alpha\beta} N} \right)^{\frac{1}{M-p}} , \tag{A.7}$$

where in the determinant we treat N as a $2p \times 2p$ matrix. A little algebra gives the equations for a supersymmetric vacuum as

$$\det[(N_{ij})^\alpha_\beta] \propto \left(\frac{\Lambda_1^{3M+p}}{\lambda^{M-p}} \right)^{2/M} \tag{A.8}$$

and

$$\lambda (N_{ij})^\alpha_\beta (N_{k\ell})^\beta_\alpha \epsilon^{ik} \epsilon^{j\ell} \propto (\lambda^p \Lambda^{3M+p})^{1/M} \,. \tag{A.9}$$

It is possible to show that these equations represent M branches, each of which is p copies of the deformed conifold — in other words, the moduli space of p probe branes moving on the deformed conifold. First, note that $N^0_{ij} \equiv (N_{ij})^\alpha_\alpha$ is a gauge invariant operator. If we demand that the SU(p)-adjoint fields $(N_{ij})^\alpha_\beta - \frac{1}{p}\delta^\alpha_\beta(N_{ij})^\gamma_\gamma$ vanish, then the equations above become

$$\det[(N^0_{ij})] \propto \left(\frac{\Lambda_1^{3M+p}}{\lambda^{M-p}}\right)^{2/M} \tag{A.10}$$

and

$$\lambda N^0_{ij} N^0_{k\ell}\epsilon^{ik}\epsilon^{j\ell} \propto (\lambda^p \Lambda^{3M+p})^{1/M} \tag{A.11}$$

which gives M branches, each of which is a single copy of the deformed conifold. This region of moduli space corresponds to taking all p probe branes to have the same positions on the conifold. As before the $\mathbb{Z}_{2M}$ global symmetry is broken to $\mathbb{Z}_2$; it is easy to see that the superpotential on the M branches rotates by a phase under the broken $\mathbb{Z}_M$. Expectation values for elements of $(N_{ij})^\alpha_\beta - \frac{1}{p}\delta^\alpha_\beta(N_{ij})^\gamma_\gamma$ correspond to moving the p probe branes apart; taking the special cases where these fields are diagonal, it is easy to show that each set of eigenvalues of $(N_{ij})^\alpha_\beta$, $i,j = 1,2$, sweeps out its own copy of the deformed conifold.

When the SU(p) gauge coupling is turned back on, the superpotential will include unknown functions of the invariants I, J, and R. These can be generated by a number of different physical phenomena, including instantons in regions where the SU(p) group is partially broken. However, as before, these functions change the quantitative features of the deformation of the conifold without altering the basic picture we have obtained. Furthermore, we expect no additional significant infrared dynamics. Above the strong-dynamics scale for SU($M+p$), the SU(p) gauge group is infrared free. Below it, the SU(p) group contains three adjoint fields $(N_{ij})^\alpha_\beta$ which have a trilinear superpotential — in short, a copy of $\mathcal{N}=4$ Yang-Mills. The SU(p) sector is therefore scale-invariant and non confining at low energy. Lastly, we expect that the SU(p) dynamics plays no role in the supergravity regime for $p \ll M$. Supergravity requires we work at small gauge coupling and large 't Hooft coupling for SU(M), but in this regime SU(p) will have small 't Hooft coupling and will be described by weakly-coupled field theory. In the end, then, we again expect M branches, given by equations of the same qualitative form as above.

The case $p = M$ is the most subtle. For SU($2M$) $\times$ SU(M), the SU($2M$) theory has equal numbers of flavors and colors, and consequently its moduli space is modified quantum mechanically [25]. If we turn off the SU(M) coupling, the superpotential becomes

$$W = \lambda (N_{ij})^\alpha_\beta (N_{k\ell})^\beta_\alpha \epsilon^{ik}\epsilon^{j\ell} F_1\left(\frac{I_1}{J_1}\right) + X\left(\det[(N_{ij})^\alpha_\beta] - \mathcal{B}\bar{\mathcal{B}} - \Lambda_{2M}^{4M}\right), \tag{A.12}$$

where the "baryon" $\mathcal{B}$ is the gauge invariant operator $A_1^M A_2^M$, and the anti-baryon is similarly constructed from B_i. Here the equations seem to have multiple solutions. One solution is

$$X = 0\,; \qquad N = 0\,; \qquad \mathcal{B} = \bar{\mathcal{B}} = i\Lambda_{2M}^{2M}\,. \tag{A.13}$$

In this case, the SU(M) gauge group is unbroken and, when its coupling is restored, it generates M distinct and isolated vacua via usual gaugino condensation. Alternatively, we may have

$$\begin{aligned} &\mathcal{B} = \bar{\mathcal{B}} = 0\,; \qquad \det[(N_{ij})^{\alpha}_{\beta}] = \Lambda_{2M}^{4M}\,; \\ &\left[(N_{ij})^{\alpha}_{\beta}(N_{k\ell})^{\beta}_{\alpha}\epsilon^{ik}\epsilon^{j\ell}G_1\left(\frac{I_1}{J_1}\right)\right]^M = \Lambda_{2M}^{4M}\,, \end{aligned} \tag{A.14}$$

where we have not determined G_1. As before this leads to M branches, each of which has M probe branes moving on a deformed conifold.

This suggests that the complete solution to a theory with gauge group SU($N+M$) $\times$ SU(N) might involve not one set of M branches but many. The smallest set would consist of $p \equiv N$ mod M D3-branes moving on the deformed conifold. The next smallest set would consist of $p+M$ D3-branes. Next would follow a branch with $p+2M$ D3-branes, and so forth, growing in size without limit. To see whether this is the case requires a more thorough and complete field theory analysis, which we have not performed.

In any case, these partial results all support the main claims of the paper: that all branches which appear are consistent with probe branes moving on a deformed conifold, and that each branch is one of M identical branches which are rotated by the spontaneously broken $\mathbb{Z}_{2M}$ R-symmetry.

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